UNIVERSAL HOPF ALGEBRA OF RENORMALIZATION AND HOPF ALGEBRAS OF ROOTED TREES

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Abstract. In this paper we are going to find a rooted tree representation from universal Hopf algebra of renormalization (in Connes-Marcolli's approach in the study of renormalizable Quantum Field Theories under the scheme minimal subtraction in dimensional regularization). With attention to this new picture, interesting relations between this specific Hopf algebra and some important Hopf algebras of rooted trees and also Hopf algebra of (quasi-) symmetric functions are obtained. And moreover a new interpretation from universal singular frame, based on Hall rooted trees, is deduced such that it can be applied to the physical information of a renormalizable theory such as counterterms.

Introduction

In modern physics the description of phenomena at the smallest length scales with highest energies is done by Quantum Field Theory (QFT) such that perturbative theory is a successful and useful approach to this powerful physical theory. Renormalization plays the main part in this attitude such that its application on the perturbative expansions of divergent iterated Feynman integrals will deliver us renormalized values together with counterterms.

With arranging Feynman diagrams of a renormalizable QFT into a Hopf algebra (such that its algebraic structures are induced based on the existence of a recursive procedure for the elimination of (sub)divergences from diagrams), Kreimer discovered a strong mathematical interpretation for the perturbative renormalization. With the concept of regularization, one can parametrize ultraviolet divergences appearing in amplitudes to reduce them formally finite together with a special subtraction of ill-defined expressions (induced with physical principles). Moreover in the process of regularization some non-physical parameters are created and this fact will change the nature of Feynman rules to algebra morphisms from the Hopf algebra (related to the given theory) to the commutative algebra of Laurent series in dimensional regularization such that in general this commutative algebra is characterized with the given regularization method [12, 13]. It provides this fact that Feynman rules of the given theory can be determined with some special characters of this Hopf algebra such that with these Feynman rules, one can associate to each Feynman graph its related amplitude. Therefore the lack of a practical mathematical basement for this interesting physical technique in QFT is covered. [1, 20, 21, 22, 23, 24, 25]

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Connes and Kreimer proved that perturbative renormalization can be explained by a general mathematical procedure namely, extraction of finite values based on the Riemann-Hilbert problem and in this way they showed that one can obtain the important physical data of a renormalizable QFT for instance renormalized values and counterterms from the Birkhoff decomposition of characters of the related Hopf algebra to the theory. In other words, they associated to each theory an infinite dimensional Lie group and proved that in dimensional regularization passing from unrenormalized to the renormalized value is equivalent to the replacement of a given loop (with values in the Lie group) with the value of the positive component of its Birkhoff factorization at the critical integral dimension D. In fact, in [2, 3] an algebraic reconstruction from the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) method in renormalization is initiated.

Connes and Marcolli completed this mathematical machinery from perturbative renormalization for renormalizable QFTs in the scheme minimal subtraction in dimensional regularization underlying the Riemann-Hilbert correspondence by giving a algebro-geometric dictionary for the description of physical theories. In addition they found a geometric manifestation from physical information of a given theory by the classification of equisingular flat connections and putting them in a category such that its objects give power to store important parameters such as counterterms. By this way, each counterterm is represented with the solution of a differential system together with a special singularity, namely equisingularity. In a general appearance, they introduced the universal category of flat equisingular vector bundles $\mathcal{E}$ such that its universality comes from this interesting notion that for each renormalizable theory Φ, one can put its related category of flat equisingular connections $\mathcal{E}^{\Phi}$ as a subcategory in $\mathcal{E}$. And because of the neutral Tannakian nature of this universal category, one specific Hopf algebra will be characterized from the process. That is universal Hopf algebra of renormalization H_U such that by this special Hopf algebra and its affine group scheme, renormalization groups and counterterms of all renormalizable physical theories will have universal and canonical lifts. [5, 6, 7]

The combinatorial nature of the Hopf algebra of renormalization is observed by its relation with the Connes-Kreimer Hopf algebra of rooted trees [20, 22]. This Hopf algebra on rooted trees has universal property with respect to the Hochschild cohomology theory [1, 23]. On the other hand, we know that the universal Hopf algebra of renormalization has universal property with respect to Hopf algebras (related to all renormalizable physical theories) [5]. These observations provide the notion of a closed relation between the elements of H_U and rooted trees.

In this work we are going to concentrate on this idea to obtain a new picture from universal Hopf algebra of renormalization by rooted trees. According to this goal, in the first section, we consider Connes-Kreimer Hopf algebra of rooted trees. In the second part, the basic definition of the universal Hopf algebra of renormalization is studied. In the third part, we familiar with some interesting Hopf algebra structures on rooted trees and also review an important technique (i.e. operadic approach) for making an explicit rooted tree interpretation from this universal element in the mathematical treatment of perturbative renormalization. In section four, we make rooted tree representations from H_U at three levels: Hopf algebra, affine group scheme and Lie algebra. Findings interesting relations between this

specific Hopf algebra and Hopf algebra of (quai-) symmetric functions and also some important Hopf algebras on rooted trees are immediate results from this rooted tree version. Finally in the last part of this work, a rooted tree representation from universal singular frame is given. Since this element maps to negative parts of the Birkhoff decomposition of loops and provides universal counterterms, therefore one can obtain a new image from counterterms.

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1. Connes-Kreimer Hopf algebra of rooted trees

Feynman diagrams together with the Feynman rules of a given renormalizable QFT present all of the possible events in the theory such that in general they contain (overlapping) sub-divergences. A rooted tree representation from these diagrams will be applied for at the first, their arranging into a Hopf algebraic structure and at the second, describing more explicitly about the process of renormalization. In the case of the overlapping divergences, at the first step they should be reduced to a linear combination of disjoint and nested divergences and next we have a sum of rooted trees [21, 26]. Depends on a fixed theory, all rooted trees are equipped with specific decorations (which store physical information such as (sub)-divergences) [12, 20, 22, 23]. Renormalization Hopf algebra of a given theory can be made based on a decorated version of an interesting Hopf algebra structure on rooted trees and in this part this Hopf algebra is studied.

A *rooted tree* t is an oriented, connected and simply connected graph together with one distinguished vertex with no incoming edge namely, root. A rooted tree t with a given embedding in the plane is called *planar rooted tree* and otherwise it is called *(non-planar) rooted tree*.

Let $\mathbf{T}$ be the set of all non-planar rooted trees and $\mathbb{K}\mathbf{T}$ be the vector space over a field $\mathbb{K}$ (with characteristic zero) generated by $\mathbf{T}$. It is graded by the number of non-root vertices of rooted trees and it means that

$$\mathbf{T_n} := \{t \in \mathbf{T} : |t| = n+1\}, \quad \mathbb{K}\mathbf{T} := \bigoplus_{n\geq 0} \mathbb{K}\mathbf{T_n}. \tag{1.1}$$

$H(\mathbf{T}) := Sym(\mathbb{K}\mathbf{T})$ is a graded free unital commutative algebra containing $\mathbb{K}\mathbf{T}$ such that the empty tree is its unit. A monomial in rooted trees (that commuting with each other) is called *forest*.

An *admissible cut* c of a rooted tree t is a collection of its edges with this condition that along any path from the root to the other vertices, it meets at most one element of c. By removing the elements of an admissible cut c from a rooted tree t, we will have a rooted tree $R_c(t)$ with the original root and a forest $P_c(t)$ of rooted trees.

This concept determines a coproduct structure on $H(\mathbf{T})$ given by

$$\Delta : H(\mathbf{T}) \longrightarrow H(\mathbf{T}) \otimes H(\mathbf{T}), \quad \Delta(t) = t \otimes \mathbb{I} + \mathbb{I} \otimes t + \sum_c P_c(t) \otimes R_c(t) \tag{1.2}$$

where the sum is over all possible non-trivial admissible cuts of t. It is observed that this coproduct can be rewritten in a recursive way. Let $B^+ : H(\mathbf{T}) \longrightarrow H(\mathbf{T})$ be a linear operator that mapping a forest to a rooted tree by connecting the roots of rooted trees in the forest to a new root. B^+ is an isomorphism of graded vector spaces and for the rooted tree $t = B^+(t_1...t_n)$, we have

$$\Delta B^+(t_1...t_n) = t \otimes \mathbb{I} + (id \otimes B^+)\Delta(t_1...t_n). \tag{1.3}$$

Δ is extended linearity to define it as an algebra homomorphism.

In addition, one can define recursively an antipode on $H(\mathbf{T})$ given by

$$S(t) = -t - \sum_c S(P_c(t))R_c(t). \tag{1.4}$$

And finally, we equip this space with the counit $\epsilon : H(\mathbf{T}) \longrightarrow \mathbb{K}$ given by

$$\epsilon(\mathbb{I}) = 1, \quad \epsilon(t_1...t_n) = 0, \quad t_1...t_n \neq \mathbb{I}. \tag{1.5}$$

$H(\mathbf{T})$ together with the coproduct (1.2) and the antipode (1.4) is a finite type connected graded commutative non-cocommutative Hopf algebra. It is called *Connes-Kreimer Hopf algebra* and denoted by H_{CK}. [1, 20, 22]

Let $H(\mathbf{T})^\star$ be the dual space that contains all linear maps from $H(\mathbf{T})$ to $\mathbb{K}$. A linear map $f : H(\mathbf{T}) \longrightarrow \mathbb{K}$ is called *character*, if

$$f(t_1t_2) = f(t_1)f(t_2), \quad f(\mathbb{I}) = 1. \tag{1.6}$$

The set of all characters is denoted by **char H(T)**. A linear map $g : H(\mathbf{T}) \longrightarrow \mathbb{K}$ is called *derivation (infinitesimal character)*, if

$$g(t_1t_2) = g(t_1)\epsilon(t_2) + \epsilon(t_1)g(t_2). \tag{1.7}$$

The set of all derivations is denoted by $\partial char H(\mathbf{T})$.

One can equip the space $H(\mathbf{T})^\star$ with the *convolution product*

$$f * g(t) := m_{\mathbb{K}}(f \otimes g)\Delta(t). \tag{1.8}$$

This product determines a group structure on **char H(T)** and a graded Lie algebra structure on $\partial char H(\mathbf{T})$ such that there is a bijection map $\exp^*$ from $\partial char H(\mathbf{T})$ to **char H(T)** (which plays an essential role in the presentation of components of the Birkhoff decomposition of characters). [9, 10, 11, 28]

Finally one should notice to the *universal property* of this Hopf algebra such that it is the result of the universal problem in Hochschild cohomology.

Theorem 1.1. *Let $\mathcal{C}$ be a category with objects (H, L) consisting of a commutative Hopf algebra H and a Hochschild one cocycle $L : H \longrightarrow H$. It means that for each $x \in H$,*

$$\Delta L(x) = L(x) \otimes \mathbb{I} + (id \otimes L)\Delta(x).$$

And also Hopf algebra homomorphisms, that commute with cocycles, are morphisms in this category. (H_{CK}, B^+) is the universal element in $\mathcal{C}$. In other words, for each object (H, L) there exists a unique morphism of Hopf algebras $\phi : H_{CK} \longrightarrow H$ such that $L \circ \phi = \phi \circ B^+$. H_{CK} is unique up to isomorphism. [1, 20]

2. Universal Hopf algebra of renormalization

The importance of this specific Hopf algebra in the mathematical treatment of perturbative renormalization is studied in [5, 6, 7] and here just we want to look at to the definition and some basic properties of this Hopf algebra. Because we would like to find a clear relation between the elements of this Hopf algebra and rooted trees. The application of this relation in Connes-Marcolli's approach will be shown in the last section.

H_U is a connected graded commutative non-cocommutative Hopf algebra of finite type. It is the graded dual of the universal enveloping algebra of the free graded Lie algebra $L_{\mathbb{U}} := \mathbf{F}(1,2,...)_{\bullet}$ generated by elements e_{-n} of degree $n > 0$ (i.e. one generator in each degree). It is interesting to know that as an algebra H_U is isomorphic to the linear space of noncommutative polynomials in variables f_n, $n \in \mathbb{N}_{>0}$ with the shuffle product [5, 6]. It is a gold key for us to find a relation between this Hopf algebra and rooted trees but at the first we need some information about shuffle structures.

Let V be a vector space over the field $\mathbb{K}$ with the tensor algebra $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$. Set

$$S(m,n) = \{\sigma \in S_{m+n} : \sigma^{-1}(1) < ... < \sigma^{-1}(m), \quad \sigma^{-1}(m+1) < ... < \sigma^{-1}(m+n)\}. \tag{2.1}$$

It is called the set of $(m,n)-$*shuffles*. For each $x = x_1 \otimes ... \otimes x_m \in V^{\otimes m}$, $y = y_1 \otimes ... \otimes y_n \in V^{\otimes n}$ and $\sigma \in S(m,n)$, define

$$\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes ... \otimes u_{\sigma(m+n)} \in V^{\otimes(m+n)} \tag{2.2}$$

such that $u_k = x_k$ for $1 \leq k \leq m$ and $u_k = y_{k-m}$ for $m+1 \leq k \leq m+n$. The *shuffle product* of x, y is given by

$$x \star y := \sum_{\sigma \in S(m,n)} \sigma(x \otimes y). \tag{2.3}$$

$(T(V), \star)$ is a unital commutative associative algebra and one can define this product in a recursive procedure. There are some extensions of this product such as *quasi-shuffles, mixable shuffles*.

Let A be a *locally finite set* (i.e. disjoint union of finite sets $A_n, n \geq 1$). The elements of A are *letters* and monomials are called *words* such that the empty word is denoted by 1. Set $A^- := A \cup \{0\}$. A locally finite set A together with a Hoffman pairing $[.,.]$ on A^- is called a *Hoffman set*. Let $\mathbb{K} < A >$ be the graded noncommutative polynomial algebra over the field $\mathbb{K}$. The *quasi-shuffle product* $\star^-$ on $\mathbb{K} < A >$ is defined recursively such that for any word w, $1 \star^- w = w \star^- 1 = w$ and also for words w_1, w_2 and letters a, b,

$$(aw_1) \star^- (bw_2) = a(w_1 \star^- bw_2) + b(aw_1 \star^- w_2) + [a,b](w_1 \star^- w_2). \tag{2.4}$$

$\mathbb{K} < A >$ together with the new product $\star^-$ is a graded commutative algebra such that when $[.,.] = 0$, it will be the shuffle algebra $(T(V), \star)$ where V is a vector space generated by the set A.

Theorem 2.1. *(i) There is a graded connected commutative non-cocommutative Hopf algebra structure (of finite type) on* $(\mathbb{K} < A >, \star^{-})$ $((T(V), \star))$ *such that its coproduct is compatible with the (quasi-)shuffle product.*

(ii) There is an isomorphism (as a graded Hopf algebras) between $(T(V), \star)$ *and* $(\mathbb{K} < A >, \star^{-})$.

(iii) There is a graded connected Hopf algebra structure (of finite type) (comes from (quasi-)shuffle product) on the graded dual of $\mathbb{K} < A >$.

(iv) We can extend the isomorphism in the second part to the graded dual level. [8, 15]

Proof. The compatible Hopf algebra structure on the shuffle algebra of noncommutative polynomials is given by the coproduct

$$\Delta(w) = \sum_{uv=w} u \otimes v \tag{2.5}$$

and the counit

$$\epsilon(1) = 1, \quad \epsilon(w) = 0, \; w \neq 1. \tag{2.6}$$

For a given Hoffman pairing $[.,.]$ and any finite sequence S of elements of the set A, with induction define $[S] \in A^{-}$ such that for any $a \in A$, $[a] = a$ and $[a, S] = [a, [S]]$. Let $C(n)$ be the set of compositions of n and $C(n, k)$ be the set of compositions of n with length k. For each word $w = a_1...a_n$ and composition $I = (i_1, ..., i_l)$, set

$$I[w] := [a_1, ..., a_{i_1}][a_{i_1+1}, ..., a_{i_1+i_2}]...[a_{i_1+...+i_{l-1}+1}, ..., a_n]. \tag{2.7}$$

It means that compositions act on words. Now for any word $w = a_1...a_n$, its antipode is given by

$$S(1) = 1,$$

$$S(w) = -\sum_{k=0}^{n-1} S(a_1...a_k) \star^{-} a_{k+1}...a_n = (-1)^n \sum_{I \in C(n)} I[a_n...a_1]. \tag{2.8}$$

To show the isomorphism between Hopf algebra structures (compatible with the shuffle products) is done by the following maps

$$\tau(w) = \sum_{(i_1,...,i_l) \in C(|w|)} \frac{1}{i_1!...i_l!} (i_1, ..., i_l)[w], \tag{2.9}$$

$$\psi(w) = \sum_{(i_1,...,i_l) \in C(|w|)} \frac{(-1)^{|w|-l}}{i_1...i_l} (i_1, ..., i_l)[w]. \tag{2.10}$$

τ is an isomorphism of Hopf algebras and ψ is its inverse.

The graded dual $\mathbb{K} < A >^{\star}$ has a basis consisting of elements $v^{\star}$ (where v is a word on A) with the following pairing such that if $u = v$, then $(u, v^{\star}) = 1$ and if $u \neq v$, then $(u, v^{\star}) = 0$. Its Hopf algebra structure is given by the concatenation product

$$conc(u^{\star} \otimes v^{\star}) = (uv)^{\star} \tag{2.11}$$

and the coproduct

$$\delta(w^{\star}) = \sum_{u,v} (u \star^{-} v, w^{\star}) u^{\star} \otimes v^{\star}. \tag{2.12}$$

The map

$$\tau^\star(u^\star) = \sum_{n\geq 1} \sum_{[a_1,...,a_n]=u} \frac{1}{n!}(a_1...a_n)^\star, \tag{2.13}$$

is an isomorphism in the dual level and its inverse is given by

$$\psi^\star(u^\star) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[a_1,...,a_n]=u} (a_1...a_n)^\star. \tag{2.14}$$

□

Let $\mathcal{L}$ be a Lie algebra over the field $\mathbb{K}$. There exists an associative algebra $\mathcal{L}_0$ over $\mathbb{K}$ together with a Lie algebra homomorphism $\phi_0 : \mathcal{L} \longrightarrow \mathcal{L}_0$ such that for each couple $(\mathcal{A}, \phi : \mathcal{L} \longrightarrow \mathcal{A})$ of an algebra and a Lie algebra homomorphism, there is a unique algebra homomorphism $\phi_{\mathcal{A}} : \mathcal{L}_0 \longrightarrow \mathcal{A}$ such that $\phi_{\mathcal{A}} \circ \phi_0 = \phi$. $\mathcal{L}_0$ is called *universal enveloping algebra* of $\mathcal{L}$ and it is unique up to isomorphism. The universal enveloping algebra $\mathcal{L}_0$ of the free Lie algebra $\mathcal{L}(A)$ is a free associative algebra on A and ϕ_0 is injective such that $\phi_0(\mathcal{L}(A))$ will be the Lie subalgebra of $\mathcal{L}_0$ generated by $\phi_0 \circ i(A)$. [30]

The set of *Lie polynomials* in $\mathbb{K} < A >^\star$ is the smallest sub-vector space of $\mathbb{K} < A >^\star$ containing the set of generators $A^\star := \{a^\star : a \in A\}$ and closed under the Lie bracket.

Corollary 2.2. *The set of Lie polynomials in $\mathbb{K} < A >^\star$ forms a Lie algebra. It is the free Lie algebra on $A^\star$ such that $\mathbb{K} < A >^\star$ is its universal enveloping algebra.*

For the given locally finite set $A = \{f_n : n \in \mathbb{N}_{>0}\}$, let V be its generated vector space. As an algebra, the universal Hopf algebra H_U is isomorphic to $(T(V), \star)$ and therefore its Hopf algebra structure is introduced by theorem 2.1. At the Lie algebra level, we have to go to the dual structure. Result 2.2 shows that the set of all Lie polynomials in $H_U^\star$ is the free Lie algebra generated by $\{f_n^\star\}_{n\in\mathbb{N}_{>0}}$ such that $H_U^\star$ is its universal enveloping algebra and on the other hand, we know that $H_U^\star$ is identified by the universal enveloping of the free graded Lie algebra $L_{\mathbb{U}}$ generated by $\{e_{-n}\}_{n\in\mathbb{N}_{>0}}$. Therefore it makes sense that

$$e_{-n} \longleftrightarrow f_n^\star. \tag{2.15}$$

3. Some interesting Hopf algebras of rooted trees

Because of the importance of a toy model from Hopf algebra of Feynman graphs for practicers in theoretical physics, rooted trees found an essential role in the study of QFT [2, 12]. There are different kinds of Hopf algebra structures on rooted trees and in this part with notice to the Connes-Kreimer renormalization, we consider a group of Hopf algebras related to the Connes-Kreimer Hopf algebra and then relations among them will be discussed.

Define a noncommutative product $\bigcirc$ on $\mathbb{K}\mathbf{T}$. Let t, s be rooted trees such that $t = B^+(t_1...t_n)$ and $|s| = m$. $t \bigcirc s$ is the sum of rooted trees given by attaching each of t_i to a vertex of s. One can define a coproduct compatible with $\bigcirc$ on $\mathbb{K}\mathbf{T}$ given by

$$\Delta_{GL} B^+(t_1, ...t_k) = \sum_{I\cup J=\{1,2,...,k\}} B^+(t(I)) \otimes B^+(t(J)). \tag{3.1}$$

$H_{GL} := (\mathbb{K}\mathbf{T}, \bigcirc, \Delta_{GL})$ is a connected graded noncommutative cocommutative Hopf algebra and it is called *Grossman-Larson Hopf algebra*. H_{GL} is the graded dual of H_{CK} and it is the universal enveloping algebra of its Lie algebra of primitives. [16, 27, 29]

Let $\mathbf{P}$ be the set of all planar rooted trees and $\mathbb{K}\mathbf{P}$ be its graded vector space. Tensor algebra $T(\mathbb{K}\mathbf{P})$ is an algebra of ordered forests of planar rooted trees and $B^+ : T(\mathbb{K}\mathbf{P}) \longrightarrow \mathbb{K}\mathbf{P}$ is an isomorphism of graded vector spaces. There are two interesting Hopf algebra structures on $\mathbf{P}$.

One can represent planar rooted trees by the symbols $<, >$ together with some special rules. It is called *balanced bracket representation or BBR*. Empty BBR is the representation of a tree with just one vertex and in the BBR of a planar rooted tree of degree n, each of the symbols $<$ and $>$ occur n times and it means that in reading from left to right, the count of $<$'s is agree with the count of $>$'s.

A BBR F is called *irreducible*, if $F =< G >$ for some BBR G and otherwise it can be written by a juxtaposition $F_1...F_k$ of irreducible BBRs. These components correspond to the branches of the root in the associated planar rooted tree.

Let t, s be two planar rooted trees with BBR representations F_t, F_s such that $F_t = F_t^1...F_t^k$. Define a new product $t \diamond s$ by a sum of planar rooted trees such that their BBRs are given by shuffling the components of F_t into the F_s. Moreover the decomposition of elements into their irreducible components determines a compatible coproduct $\Delta_\diamond$ on $\mathbb{K}\mathbf{P}$. By these new operations (given based on the balanced bracket representation), there is a connected graded noncommutative Hopf algebra structure on $\mathbb{K}\mathbf{P}$ and it is denoted by $H_{\mathbf{P}} := (\mathbb{K}\mathbf{P}, \diamond, \Delta_\diamond)$. [17]

Another important Hopf algebra structure on planar rooted trees is defined on the tensor algebra $T(\mathbb{K}\mathbf{P})$ together with the given coproduct structure in (1.2) (on planar rooted trees instead of rooted trees) such that it gives a graded connected noncommutative Hopf algebra. It is called *Foissy Hopf algebra* and denoted by H_F. H_F is self-dual and isomorphic to $H_{\mathbf{P}}$. [16, 17, 18]

Let $\mathbb{K}[[x_1, x_2, ...]]$ be the ring of formal power series. A formal series f is called *symmetric (quasi-symmetric)*, if the coefficients in f of the monomials $x_{n_1}^{i_1}...x_{n_k}^{i_k}$ and $x_1^{i_1}...x_k^{i_k}$ equal for any sequence of distinct positive integers $n_1, ..., n_k$ (for any increasing sequence $n_1 < ... < n_k$). In other words, we know that the symmetric groups $\mathbb{S}_n$ act on $\mathbb{K}[[x_1, x_2, ...]]$ by permuting the variables and a symmetric function is invariant under these actions and it means that after each permutation coefficients of its monomials remain without any change. [14]

Let SYM ($QSYM$) be the set of all symmetric (quasi-symmetric) functions. It is easy to see that $SYM \subset QSYM$. As a vector space, $QSYM$ is generated by the monomial quasi-symmetric functions M_I such that $I = (i_1, ..., i_k)$ and $M_I := \sum_{n_1 < n_2 < ... < n_k} x_{n_1}^{i_1}...x_{n_k}^{i_k}$ and if we forget order in a composition, then we will reach to the generators $m_\lambda := \sum_{\phi(I)=\lambda} M_I$ for SYM (viewed as a vector space). [17, 18]

One can show that there is a graded connected commutative cocommutative self-dual Hopf algebra structure on SYM and also there is a graded connected commutative non-cocommutative Hopf algebra structure on $QSYM$ such that its graded dual is denoted by $NSYM$. As an algebra, $NSYM$ is the noncommutative polynomials on the variables z_n of degree n. [14, 17, 18]

(Quasi-)symmetric functions play an interesting role to identify some relations between this type of functions and various defined Hopf algebras on (planar) rooted trees.

Theorem 3.1. *There are following commutative diagrams of Hopf algebra homomorphisms.* [17]

$$\begin{array}{ccc} NSYM & \xrightarrow{\alpha_1} & H_F \\ {\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\alpha_2}\downarrow \\ SYM & \xrightarrow{\alpha_4} & H_{CK} \end{array} \tag{3.2}$$

$$\begin{array}{ccc} SYM & \xleftarrow{\alpha_4^\star} & H_{GL} \\ {\scriptstyle\alpha_3^\star}\downarrow & & {\scriptstyle\alpha_2^\star}\downarrow \\ QSYM & \xleftarrow{\alpha_1^\star} & H_{\mathbf{P}} \end{array} \tag{3.3}$$

Proof. - α_1 associates each variable z_n to the ladder tree l_n of degree n.

- α_2 maps each planar rooted tree to its corresponding rooted tree without notice to the order in products.

- α_3 sends each z_n to the symmetric function $m_{\underbrace{(1,...,1)}_{n}}$.

- α_4 maps $m_{\underbrace{(1,...,1)}_{n}}$ to the ladder tree l_n.

- For the composition $I = (i_1, ..., i_k)$ one can define a planar rooted tree $t_I := B^+(l_{i_1}, ..., l_{i_k})$. For each planar rooted tree t, if $t = t_I$, then define $\alpha_1^\star(t) := M_I$ and otherwise $\alpha_1^\star(t) := 0$.

- For each rooted tree t, $\alpha_2^\star(t) := |sym(t)| \sum_{s \in \alpha_2^{-1}(t)} s$.

- $\alpha_3^\star$ is the inclusion map.

- For the partition $J = (j_1, ..., j_k)$, define a rooted tree $t_J := B^+(l_{j_1}, ..., l_{j_k})$. For each rooted tree t, if $t = t_J$ (for some partition J), then define $\alpha_4^\star(t) := |sym(t_J)| m_J$ and otherwise $\alpha_4^\star(t) := 0$.

□

Zhao's homomorphism provides another important group of relations between rooted trees and (quasi-)symmetric functions. This map is defined by

$$Z : NSYM \longrightarrow H_{GL}, \quad Z(z_n) = \epsilon_n \tag{3.4}$$

such that rooted trees ϵ_n are given recursively by

$$\epsilon_0 := \bullet$$

$$\epsilon_n := k_1 \bigcirc \epsilon_{n-1} - k_2 \bigcirc \epsilon_{n-2} + ... + (-1)^{n-1} k_n \tag{3.5}$$

where

$$k_n := \sum_{|t|=n+1} \frac{t}{|sym(t)|} \in H_{GL}. \tag{3.6}$$

Z is an injective homomorphism of Hopf algebras and its dual can be clarified uniquely. Define a linear map

$$A^{+} : QSYM \longrightarrow QSYM, \quad M_I \longmapsto M_{I \sqcup (1)}. \tag{3.7}$$

For each ladder tree l_n and monomial u of rooted trees, the surjective homomorphism $Z^{\star} : H_{CK} \longrightarrow QSYM$ is given by

$$Z^{\star}(l_n) := M_{\underbrace{(1,...,1)}_{n}},$$

$$Z^{\star}(B^{+}(u)) := A^{+}(Z^{\star}(u)). \tag{3.8}$$

With the cocycle property of the map A^{+}, one can show that $Z^{\star}$ is the unique homomorphism with respect to the relation (3.8) [18, 19]. In the next section, we will lift the Zhao's homomorphism and its dual to the level of the universal Hopf algebra of renormalization.

Operadic approach to the poset theory provides an interesting source of Hopf algebras namely, incidence Hopf algebras such that because of the relation between Connes-Kreimer Hopf algebra and this class of Hopf algebras, one can find the application of this part of mathematics in the study of perturbative renormalization. In the final part of this section this theory will be considered.

A partially ordered set (*poset*) is a set with a partial order relation. A growing sequence of the elements of a poset is called *chain*. A poset is *pure*, if for any $x \leq y$ the maximal chains between x and y have the same length. A bounded and pure poset is called *graded poset*. For instance one can define a graded partial order on the set $[n] = \{1, 2, ..., n\}$ by the refinement of partitions and it is called *partition poset*.

A collection $\{P(n)\}_n$ of (right) $\mathbb{S}_n-$modules is called $\mathbb{S}-$*module*. An *operad* (P, μ, η) is a monoid in the monoidal category $\mathbb{S} - Mod := (\mathbb{S} - Mod, \circ, \mathbb{I})$ of $\mathbb{S}-$modules and it means that the composition morphism $\mu : P \circ P \longrightarrow P$ is associative and the morphism $\eta : \mathbb{I} \longrightarrow P$ is unit. This operad is called *augmented*, if there exists a morphism of operads $\psi : P \longrightarrow \mathbb{I}$ such that $\psi \circ \eta = id$.

A $\mathbb{S}-$*set* is a collection $\{P_n\}_n$ of sets P_n equipped with an action of the group $\mathbb{S}_n$. A monoid (P, μ, η) in the monoidal category of $\mathbb{S}-$sets is called a *set operad*. For each $(x_1, ..., x_t) \in P_{i_1} \times ... \times P_{i_t}$, one can define the map

$$\lambda_{(x_1,...,x_t)} : P_t \longrightarrow P_{i_1+...+i_t}$$

$$x \longmapsto \mu(x \circ (x_1, ..., x_t)). \tag{3.9}$$

A set opeard P is called *basic*, if each $\lambda_{(x_1,...,x_t)}$ is injective.

Partition posets associated to an operad is interesting for us. For the set operad (P, μ, η) and each n, there is an action of the group $\mathbb{S}_n$ on P_n. For the set A with n elements, let $\mathbb{A}$ be the set of ordered sequences of the elements of A such that each element appearing once. For each element $x_n \times (a_{i_1}, ..., a_{i_n})$ in $P_n \times \mathbb{A}$, its image under an element σ of $\mathbb{S}_n$ is given by $\sigma(x_n) \times (a_{\sigma^{-1}(i_1)}, ..., a_{\sigma^{-1}(i_n)})$. It is called *diagonal action* and its orbit is denoted by $\overline{x_n \times (a_{i_1}, ..., a_{i_n})}$. Let $\mathfrak{P}_n(A) := P_n \times_{\mathbb{S}_n} \mathbb{A}$ be the set of all orbits under this action. Set

$$P(A) := (\bigsqcup_{f:[n] \longrightarrow^{bijection} A} P_n)_{\sim} \tag{3.10}$$

where $(x_n, f) \sim (\sigma(x_n), f \circ \sigma^{-1})$ is an equivalence relation. A $P-$*partition* of $[n]$ is a set of components $B_1, ..., B_t$ such that each B_j belongs to $\mathfrak{P}_{i_j}(I_j)$ where $i_1 + ... + i_t = n$ and $\{I_j\}_{1 \leq j \leq t}$ is a partition of $[n]$. We can extend maps $\lambda_{(x_1,...,x_t)}$ to $\lambda^{\sim}$ at the level of $P(A)$ and it means that

$$\lambda^{\sim} : P_t \times (\mathfrak{P}_{i_1}(I_1) \times ... \times \mathfrak{P}_{i_t}(I_t)) \longrightarrow \mathfrak{P}_{i_1+...+i_t}(A)$$

$$x \times (c_1, ..., c_t) \longmapsto \overline{\mu(x \circ (x_1, ..., x_t)) \times (a_1^1, ..., a_{i_t}^t)} \tag{3.11}$$

such that $\{I_j\}_{1 \leq j \leq t}$ is a partition of A and each c_r is represented by $\overline{x_r \times (a_1^r, ..., a_{i_r}^r)}$ where $x_r \in P_{i_r}$, $I_r = \{a_1^r, ..., a_{i_r}^r\}$.

For the set operad P and $P-$partitions $\mathfrak{B} = \{B_1, ..., B_r\}$, $\mathfrak{C} = \{C_1, ..., C_s\}$ of $[n]$ such that $B_k \in \mathfrak{P}_{i_k}(I_k)$ and $C_l \in \mathfrak{P}_{j_l}(J_l)$, we say that the $P-$partition $\mathfrak{C}$ is larger than $\mathfrak{B}$, if for any $k \in \{1, 2, ..., r\}$ there exists the subset $\{p_1, ..., p_t\} \subset \{1, 2, ..., s\}$ such that $\{J_{p_1}, ..., J_{p_t}\}$ is a partition of I_k and if there exists an element $x_t \in P_t$ such that $B_k = \lambda^{\sim}(x_t \times (C_{p_1}, ..., C_{p_t}))$. This poset is called *operadic partition poset* associated to the operad P and denoted by $\Pi_P([n])$. [32, 33]

One can extend the notion of this poset to each locally finite set $A = \bigsqcup A_n$ such that in this case a $P-$partition of $[A]$ is a disjoint union (composition) of $P-$partitions of $[A_n]$s and therefore the operadic partition poset associated to the operad P will be a composition of posets $\Pi_P([A_n])$ and denoted by $\Pi_P([A])$.

A collection $(\mathfrak{p}_i)_{i \in I}$ of posets is called *good collection*, if each poset $\mathfrak{p}_i$ has a minimal element $\mathbf{0}$ and a maximal element $\mathbf{1}$ (an interval) and also for all $x \in \mathfrak{p}_i$ $(i \in I)$, the interval $[\mathbf{0}, x]$ $([x, \mathbf{1}])$ is isomorphic to a product of posets $\prod_j \mathfrak{p}_j$ $(\prod_k \mathfrak{p}_k)$.

For a given good collection $\mathcal{A} := (\mathfrak{p}_i)_{i \in I}$, it is possible to make a new good collection $\mathcal{A}^-$ of all finite products $\prod_i \mathfrak{p}_i$ of elements such that it is closed under products and closed under taking subintervals. Let $[\mathcal{A}]$ $([\mathcal{A}^-])$ be the set of isomorphism classes of posets in $\mathcal{A}$ $(\mathcal{A}^-)$ such that elements in these sets denoted by $[i], [j], ...$ and $H_{\mathcal{A}}$ be a vector space generated by the elements $\{F_{[i]}\}_{[i] \in [\mathcal{A}^-]}$. It is equipped with a commutative product (i.e. direct product of posets) $F_{[i]} F_{[j]} = F_{[i \times j]}$ such that $F_{[e]}$ is the unit (where $[e]$ is the isomorphism class of the singleton interval). One should notice that as an algebra $H_{\mathcal{A}}$ may not be free. By the concept of subinterval, there is a coproduct structure on $H_{\mathcal{A}}$ given by

$$\Delta(F_{[i]}) = \sum_{x \in \mathfrak{p}_i} F_{[\mathbf{0},x]} \otimes F_{[x,\mathbf{1}]}. \tag{3.12}$$

With the coproduct (3.12), $H_{\mathcal{A}}$ has a commutative Hopf algebra structure.

Theorem 3.2. *Let Π_P be a family of the operadic partition posets associated to the set operad P. There is a good collection of posets $(\mathfrak{p}_i)$ (determined with Π_P). Its associated Hopf algebra H_P is called incidence Hopf algebra.* [4, 32]

One important note is that H_P has a basis indexed by isomorphism classes of intervals in the posets $\Pi_P(I)$ (for all sets I) and this identification makes the sets I disappear and it means that the construction of this Hopf algebra is independent of any label.

A rooted tree looks like a poset with a unique minimal element (root) such that for any element v, the set of elements descending v forms a chain (i.e. the graph has no loop) and maximal elements are called *leaves*. There is an interesting basic

set operad on rooted trees such that we will see that its incidence Hopf algebra is isomorphic to the Connes-Kreimer Hopf algebra.

For the set I with the partition $\{J_i\}_{i\geq 1}$, suppose $NAP(I)$ be the set of rooted trees with vertices labeled by I. For $s_i \in NAP(J_i)$ and $t \in NAP(I)$, we consider the disjoint union of the rooted trees s_i such that for each edge of t between i_1, i_2 in I, add an edge between the root of s_{i_1} and the root of s_{i_2}. The resulting graph is a rooted tree labeled by $\bigsqcup_i J_i$ and its root is the root of s_k such that k is the label of the root of t. It gives us the composition $t((s_i)_{i\in I})$. In a more general setting, it is observed that the operad NAP plays the role of a functor from the groupoid of sets to the category of sets. The operadic partition poset $\Pi_{NAP}(I)$ is a set of forests of $I-$labeled rooted trees such that a forest x is covered by a forest y, if y is obtained from x by grafting the root of one component of x to the root of another component of x. Or x is obtained from y by removing an edge incident to the root of one component of y.

Any interval in $\Pi_{NAP}(I)$ is a product of intervals of the form $[\mathbf{0}, t_i]$ such that $t_i \in NAP(J_i)$. If $t = B^+(t_1...t_k)$, then the poset $[\mathbf{0}, t]$ is isomorphic to the product of the posets $[\mathbf{0}, B^+(t_i)]$ for $i \in \{1, 2, ..., k\}$.

The incidence Hopf algebra H_{NAP} is a free commutative algebra on unlabeled rooted trees of root-valence 1 such that elements $F_{[t]}$ (where t is a rooted tree) form a basis at the vector space level. According to 1.1 and the structure of H_{NAP}, one can obtain the next important fact.

Theorem 3.3. *H_{NAP} is isomorphic to H_{CK} by the unique Hopf algebra isomorphism $\rho : F_{[B^+(t_1...t_k)]} \longmapsto t_1...t_k$.* [4]

By this result, an operadic picture from Connes-Kreimer Hopf algebra of rooted trees is given such that it will be applied to obtain an operadic representation from the universal Hopf algebra of renormalization.

4. Rooted tree version of the universal Hopf algebra of renormalization

A Hopf algebra structure is hidden in the process of perturbative renormalization such that its description is possible with help of the Hopf algebra H_{CK} of rooted trees and in fact, rooted trees (equipped with decorations related to a given theory) play the role of a simplified model. On the other hand, the universal affine group scheme $\mathbb{U}$ governs the structure of divergences of all renormalizable theories and the universality of H_U comes back to its independency from all theories [5, 7]. In this section, we want to provide an explicit interpretation from H_U by rooted trees at three different levels Hopf algebra, Lie algebra and affine group scheme. With the help of this new representation, we will obtain interesting relations between Hopf algebra H_U and other defined Hopf algebras in the previous parts.

There is a natural partial order $\preceq$ on the set of all rooted trees $\mathbf{T}$. We say $t \preceq s$, if t can be obtained from s by removing some non-root vertices and edges and it implies that $|t| \leq |s|$. From now let $\mathbf{T}(\mathbf{A})$ ($\mathbf{F}(\mathbf{A})$) be the set of all rooted trees (forests) labeled by the set A.

For $a \in A$, $t_1, ..., t_m \in \mathbf{T}(\mathbf{A})$ such that $u = t_1...t_m \in \mathbf{F}(\mathbf{A})$, $B^+_a(u)$ is a labeled rooted tree of degree $|t_1| + ... + |t_m| + 1$ obtained by grafting the roots of $t_1, ..., t_m$ to a new root labeled by a. It is clear that $B^+_a(\mathbb{I})$ is a rooted tree with just one labeled vertex.

For $t \in \mathbf{T}(\mathbf{A})$ and $u \in \mathbf{F}(\mathbf{A})$, define a new element $t \circ u$ such that it is a labeled rooted tree of degree $|t| + |u|$ given by grafting the roots of labeled rooted trees in u to the root of t.

Remark 4.1. (i) $\circ$ is not associative.

(ii) $\forall t \in \mathbf{T}(\mathbf{A})$, $\forall u, v \in \mathbf{F}(\mathbf{A}) : (t \circ u) \circ v = t \circ (uv) = (t \circ v) \circ u$.

(iii) $t_1 \circ ... \circ t_m \circ u = t_1 \circ (t_2 \circ ... \circ (t_m \circ u))$, $\quad t^{\circ k} = t \circ ... \circ t$, k times.

(iv) For each $u \in \mathbf{F}(\mathbf{A})$, let $per(u)$ be the number of different permutations of the vertices of a labeled partially ordered set that representing u. Then

$$per(\mathbb{I}) = 1, \quad per(B_a^+(u)) = per(u).$$

And if $u = \prod_{j=1}^m (t_j)^{i_j}$, then

$$per(u) = \prod_{j=1}^m i_j! per(t_j)^{i_j}.$$

(v) The bilinearity extension of $\circ$ to the linear combinations of labeled rooted trees (linear combinations of labeled forests) is also possible.

Definition 4.2. A set $\mathbf{H}(\mathbf{T}(\mathbf{A}))$ of labeled rooted trees is called *Hall set*, if it has following conditions:

- There is a total order relation $>$ on $\mathbf{H}(\mathbf{T}(\mathbf{A}))$.
- If $a \in A$, then $B_a^+(\mathbb{I}) \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$.
- For $a \in A$, $u \in \mathbf{F}(\mathbf{A}) - \{\mathbb{I}\}$ such that $u = t_1^{\circ r_1} ... t_m^{\circ r_m}$, $t_1, ..., t_m \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$, $r_1, ..., r_m \geq 1$, $t_1 > ... > t_m$,

$$B_a^+(u) \in \mathbf{H}(\mathbf{T}(\mathbf{A})) \Longleftrightarrow t_m > B_a^+(t_1^{\circ r_1} ... t_{m-1}^{\circ r_{m-1}}) \in \mathbf{H}(\mathbf{T}(\mathbf{A})).$$

- If $t = B_a^+(t_1^{\circ r_1} ... t_m^{\circ r_m}) \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$ such that $t_1, ..., t_m \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$, $r_1, ..., r_m \geq 1$, $a \in A$, then for each $j = 1, ..., m$, $t_j > t$.

From definition, for each $t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$, $r \geq 1$ and $a \in A$, it is easy to see that

$$B_a^+(t^{\circ r}) \in \mathbf{H}(\mathbf{T}(\mathbf{A})) \Longleftrightarrow t > B_a^+(\mathbb{I}). \tag{4.1}$$

For the Hall set $\mathbf{H}(\mathbf{T}(\mathbf{A}))$, the set of its forests is given by

$$\mathbf{H}(\mathbf{F}(\mathbf{A})) := \{\mathbb{I}\} \cup \{t_1^{r_1} ... t_m^{r_m} : r_1, ..., r_m \geq 1, t_1, ..., t_m \in \mathbf{H}(\mathbf{T}(\mathbf{A})), t_i \neq t_j (i \neq j)\}. \tag{4.2}$$

The relation between the elements of $\mathbf{H}(\mathbf{F}(\mathbf{A}))$ and rooted trees is clarified by the map

$$\xi : \mathbf{H}(\mathbf{F}(\mathbf{A})) - \{\mathbb{I}\} \longrightarrow \mathbf{T}(\mathbf{A})$$

$$t_1^{\circ r_1} ... t_m^{\circ r_m} \longrightarrow t_1^{\circ r_1} \circ (t_2^{\circ r_2} ... t_m^{\circ r_m}). \tag{4.3}$$

ξ is injective and its image is the set $\{B_a^+(u) \in \mathbf{T}(\mathbf{A}) : u \in \mathbf{H}(\mathbf{F}(\mathbf{A})), a \in A\}$. Hall trees and Hall forests have no symmetry. There is a one to one correspondence between a Hall set of $A-$labeled rooted trees and a Hall set of words on A. [30]

For $t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$, there is a *standard decomposition* $(t^1, t^2) \in \mathbf{H}(\mathbf{T}(\mathbf{A})) \times \mathbf{H}(\mathbf{T}(\mathbf{A}))$ such that

- If $|t| = 1$, the decomposition is $t^1 = t$, $t^2 = \mathbb{I}$,
- And if $t = B_a^+(t_1^{\circ r_1} ... t_m^{\circ r_m})$ such that $r_1, ..., r_m \geq 1$, $t_1, ..., t_m \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$: $t_1 > ... > t_m$, $a \in A$, the decomposition is given by

$$t^1 = B_a^+(t_1^{\circ r_1} ... t_{m-1}^{\circ r_{m-1}} t_m^{\circ r_m - 1}), \quad t^2 = t_m. \tag{4.4}$$

- For a Hall forest $u \in \mathbf{H}(\mathbf{F}(\mathbf{A})) - \mathbf{H}(\mathbf{T}(\mathbf{A}))$ such that $u = t_1^{\circ r_1}...t_m^{\circ r_m}$, $t_1 > ... > t_m$, the decomposition is given by $(u^1, u^2) \in \mathbf{H}(\mathbf{F}(\mathbf{A})) \times \mathbf{H}(\mathbf{T}(\mathbf{A}))$ where

$$u^1 = t_1^{\circ r_1}...t_{m-1}^{\circ r_{m-1}} t_m^{\circ r_m - 1}, \quad u^2 = t_m. \tag{4.5}$$

For a given map that associates to each word w on A a scalar $\alpha_w \in \mathbb{K}$, define a map $\alpha : \mathbf{F}(\mathbf{A}) \longrightarrow \mathbb{K}$ such that

- $\mathbb{I} \longmapsto \alpha_1$.

- For each $u \in \mathbf{F}(\mathbf{A}) - \{\mathbb{I}\}$, there is a labeled partially ordered set $(\mathfrak{u}(A), \geq)$ that represents the forest u such that vertices $x_1, ..., x_n, ...$ of this poset are labeled by $l(x_i) = a_i \in A$ $(1 \leq i)$. Let $>_{\mathfrak{u}(A)}$ be a total order relation on the set of vertices $\mathfrak{u}(A)$ such that it is an extension of the partial order relation $\geq$ on $\mathfrak{u}(A)$. For each ordered sequence $x_{i_1} >_{\mathfrak{u}(A)} ... >_{\mathfrak{u}(A)} x_{i_n}$ in $\mathfrak{u}(A)$, its corresponding word $a_{i_1}...a_{i_n}$ is denoted by $w(>_{\mathfrak{u}(A)})$. Set

$$\alpha(u) := \sum_{>_{\mathfrak{u}(A)}} \alpha_{w(>_{\mathfrak{u}(A)})}, \tag{4.6}$$

where the sum is over all total order relations $>_{\mathfrak{u}(A)}$ (i.e. extensions of the main partial order relation $\geq$) on the set of vertices of $\mathfrak{u}(A)$.

One can define another map

$$\pi : \mathbb{K}[\mathbf{T}(\mathbf{A})] \longrightarrow (\mathbb{K} < A >, \star^-), \quad \pi(u) := \sum_{>_{\mathfrak{u}(A)}} w(>_{\mathfrak{u}(A)}) \tag{4.7}$$

such that for each $u, v \in \mathbf{F}(\mathbf{A})$ and $a \in A$, it is observed that

$$\pi(B_a^+(u)) = \pi(u)a, \quad \pi(uv) = \pi(u) \star^- \pi(v), \quad \alpha(u) = \widehat{\alpha}(\pi(u)) \tag{4.8}$$

where $\widehat{\alpha} : (\mathbb{K} < A >, \star^-) \longrightarrow \mathbb{K}$, $\widehat{\alpha}(w) = \alpha_w$ is a $\mathbb{K}-$linear map.

There is a canonical map f on Hall rooted trees defined by $f(a) = a$, if $a \in A$ and $f(t) = f(t^1)f(t^2)$, if t is of degree ≥ 2 with the standard decomposition $t = (t^1, t^2)$. The function f is called *foliage* and for each Hall tree t, its degree $|f(t)|$ is the number of leaves of t. The foliage of a Hall tree is called *Hall word* and for each word w on A, there is a unique factorization $w = f(t_1)...f(t_n)$ such that $t_i \in \mathbf{H}(\mathbf{T}(\mathbf{A}))$ and $t_1 > ... > t_n$ and also one can show that Hall sets of labeled rooted trees can be reconstructed recursively from an arbitrary Hall set of words on A. [30]

Let A be a totally ordered set. The *alphabetical ordering* gives a total order on the set of words on A such that for any nonempty word v, put $u < uv$ and also for letters $a < b$ and words w_1, w_2, w_3, put $w_1 a w_2 < w_1 b w_3$. A non-trivial word w is called *Lyndon*, if for any non-trivial factorization $w = uv$, we have $w < v$. The set of Lyndon words, ordered alphabetically, is a Hall set. We want to show that the consideration of its corresponding Hall set of labeled rooted trees is interested.

Theorem 4.3. *Let A be a locally finite set with a total order relation on it. The (quasi-)shuffle algebra $(\mathbb{K} < A >, \star^-)$ is the free polynomial algebra on the Lyndon words.* [15]

We know that the universal Hopf algebra of renormalization (as an algebra) is given by the shuffle product on the linear space of noncommutative polynomials with variables f_n $(n \in \mathbb{N})$. With notice to the correspondence (2.15), one can define a natural total order relation (depended on degrees of the generators e_{-n}, $(n \in \mathbb{N})$ of the free Lie algebra $L_{\mathbb{U}}$) on the set $A = \{f_n : n \in \mathbb{N}_{>0}\}$. It is given by

$$f_n < f_m \Longleftrightarrow n > m. \tag{4.9}$$

Now 4.3 shows that H_U (as an algebra) should be free polynomial algebra of the Lyndon words on the set A. Consider the Hall set of these Lyndon words (ordered alphabetically) such that its corresponding Hall set of labeled rooted trees is denoted by $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$. It makes sense to say that Lyndon words play the role of a bridge between rooted trees and H_U.

Let us consider free commutative algebra $\mathbb{K}[\mathbf{T(A)}]$ such that the set $\{t_1^{r_1}...t_m^{r_m} : t_1,...,t_m \in \mathbf{T(A)}\}$ is a $\mathbb{K}-$basis (as a graded vector space) where each expression $t_1^{r_1}...t_m^{r_m}$ is a forest. For the forest u with the associated partial order set $(\mathfrak{u}(A), \geq)$, its coproduct is given by

$$\Delta(u) = \sum_{(\mathfrak{v}(A),\mathfrak{w}(A)) \in R(\mathfrak{u}(A))} v \otimes w \tag{4.10}$$

such that labeled forests v, w are represented by labeled partially ordered subsets $\mathfrak{v}(A), \mathfrak{w}(A)$ of $\mathfrak{u}(A)$ together with the following properties:

- The set of vertices in $\mathfrak{u}(A)$ is the disjoint union of the set of vertices $\mathfrak{v}(A)$ and $\mathfrak{w}(A)$,

- For each $x, y \in \mathfrak{u}(A)$ such that $x \geq y$, if $x \in \mathfrak{w}(A)$ then $y \in \mathfrak{w}(A)$.

By (4.10), there is a connected graded commutative Hopf algebra structure on $\mathbb{K}[\mathbf{T(A)}]$ such that the product in the dual space $\mathbb{K}[\mathbf{T(A)}]^\star$ corresponds to the given coproduct (4.10) namely, dual to the coalgebra structure and it means that for each $\alpha, \beta \in \mathbb{K}[\mathbf{T(A)}]^\star$ and each forest u,

$$< \alpha\beta, u > = < \alpha \otimes \beta, \Delta(u) > . \tag{4.11}$$

One can show that H_{GL} (labeled by the set A) and $\mathbb{K}[\mathbf{T(A)}]$ are graded dual to each other and moreover with respect to the coproduct structure (4.10) and the operator B_a^+, this Hopf algebra has the universal property.

Theorem 4.4. *Let H be a commutative Hopf algebra over the field $\mathbb{K}$ and $\{L_a : H \longrightarrow H\}$ be a family of $\mathbb{K}-$linear maps such that $\cup_{a\in A} Im L_a \subset ker\epsilon_H$ and $\Delta_H L_a(c) = L_a(c) \otimes \mathbb{I}_H + (id_H \otimes L_a)\Delta_H(c)$. Then there exists a unique Hopf algebra homomorphism $\psi : \mathbb{K}[\boldsymbol{T(A)}] \longrightarrow H$ such that for each $u \in \mathbb{K}[\boldsymbol{T(A)}]$ and $a \in A$, $\psi(B_a^+(u)) = L_a(\psi(u))$.* [27]

If we look at to 1.1 (universal property of the Connes-Kreimer Hopf algebra), then it is easy to understand that 4.4 is a poset version of this universal object and it means that H_{CK} (labeled by the set A) is isomorphic to $\mathbb{K}[\mathbf{T(A)}]$.

Now one can observe that the $\mathbb{K}-$linear map π is a Hopf algebra homomorphism and for each $a_1, ..., a_m \in A$, we have

$$\pi(B_{a_m}^+ ... B_{a_2}^+(a_1)) = a_1...a_m. \tag{4.12}$$

It provides a bijection between the set of non-empty words and the set of labeled rooted trees without side-branchings. This shows that π is an epimorphism and for each $\widehat{\alpha}, \widehat{\beta} \in \mathbb{K} < A >^\star$, $u \in \mathbb{K}[\mathbf{T(A)}]$, we have

$$< \widehat{\alpha}\widehat{\beta}, \pi(u) > = < \alpha\beta, u > \tag{4.13}$$

such that $\alpha, \beta \in \mathbb{K}[\mathbf{T(A)}]^\star$. On the other hand, $I := Ker\pi$ is a Hopf ideal in $\mathbb{K}[\mathbf{T(A)}]$ and there is an explicit picture from its generators given by

$$I = Ker\pi = \tag{4.14}$$

$< \{\prod_{i=1}^{m} t_i - \sum_{i=1}^{m} t_i \circ \prod_{j\neq i} t_j : m > 1, t_1, ..., t_m \in \mathbf{T(A)}\} >=< \{t \circ z + z \circ t - tz : t, z \in \mathbf{T(A)}\} \cup \{s \circ t \circ z + s \circ z \circ t - s \circ (tz) : t, z, s \in \mathbf{T(A)}\} >=< \{t \circ z + z \circ t - tz : t, z \in \mathbf{T(A)}\} \cup \{s \circ (tz) + z \circ (ts) + t \circ (sz) - tzs : t, z, s \in \mathbf{T(A)}\} >$.

Theorem 4.5. *The (quasi-)shuffle Hopf algebra* $(\mathbb{K} < A >, \star^{-})$ *is isomorphic to the quotient Hopf algebra* $\frac{\mathbb{K}[\boldsymbol{T(A)}]}{I}$. *As an* $\mathbb{K}-$*algebra,* $\frac{\mathbb{K}[\boldsymbol{T(A)}]}{I}$ *is freely generated by the set* $\{t + I : t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))\}$. [27]

According to the above theorem and also the given operadic picture from Connes-Kreimer Hopf algebra in the previous section, we can reach to the following result.

Corollary 4.6. *(i) For each locally finite set* A *together with a total order relation, there exist Hopf ideals* J_1, J_2 *such that*

$$(\mathbb{K} < A >, \star^{-}) \cong \frac{\mathbb{K}[\boldsymbol{T(A)}]}{I} \cong \frac{H_{CK}(A)}{J_1} \cong \frac{H_{NAP}(A)}{J_2}.$$

(ii) Universal Hopf algebra of renormalization is isomorphic to a quotient of the (labeled) incidence Hopf algebra with respect to the basic set operad NAP.

Proof. By 3.3, 4.5, it is enough to set $J_1 := I$ and $J_2 := \rho^{-1} J_1$. 4.5 is a representation of H_U by rooted trees. It is enough to replace the set A with the variables f_n such that the identified Lyndon words (with the shuffle structure of H_U) gives us the Hall set $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$. □

One can obtain interesting relations between the introduced Hopf algebras in the previous parts and H_U.

Proposition 4.7. *We have following commutative diagrams of Hopf algebra homomorphisms.*

$$\begin{array}{ccc} NSYM & \xrightarrow{\beta_1} & H_F \\ {\scriptstyle\beta_3}\downarrow & & {\scriptstyle\beta_2}\downarrow \\ SYM & \xrightarrow{\beta_4} & H_U \end{array} \tag{4.15}$$

$$\begin{array}{ccc} SYM & \xleftarrow{\beta_4^{\star}} & U(L_{\mathbb{U}}) \\ {\scriptstyle\beta_3^{\star}}\downarrow & & {\scriptstyle\beta_2^{\star}}\downarrow \\ QSYM & \xleftarrow{\beta_1^{\star}} & H_{\mathbf{P}} \end{array} \tag{4.16}$$

Proof. For each (planar) rooted tree t, one can put different labels (with elements of the locally finite set $A = \{f_n\}_n$) on its vertices. Let $[t]$ be the class of all different possible Hall (planar) rooted trees with respect to t in $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$ and for the forest u, let $[u]$ be the class of all different Hall forests associated to u in $\mathbf{H}(\mathbf{F}(\mathbf{A}))_U$. With notice to the diagrams in 3.1 and with help of the given results in 4.5 and 4.6, define

- $\beta_1 := \alpha_1$,
- $\beta_3 := \alpha_3$,
- For each forest u of planar rooted trees, $\beta_2(u) := \sum_{v\in[u]} \pi(v)$,
- For each symmetric function $m_{\underbrace{(1,...,1)}_{n}}$, $\beta_4(m_{\underbrace{(1,...,1)}_{n}}) := \sum_{v\in[l_n]} \pi(v)$,
- $\beta_1^{\star} := \alpha_1^{\star}$,
- $\beta_3^{\star} := \alpha_3^{\star}$,

- For each generator e_{-n}, $\beta_4^\star(e_{-n}) := \alpha_4^\star(l_n)$,
- For each generator e_{-n}, $\beta_2^\star(e_{-n}) := \alpha_2^\star(l_n)$.

□

Now we want to lift the Zhao's homomorphism (3.4) and its dual to the level of the universal Hopf algebra of renormalization. We know that π is a surjective homomorphism from $H_{CK}(A)$ to H_U and on the other hand, $Z^\star$ gives us a unique surjective map from H_{CK} to $QSYM$ with the property (3.8). For a word w with length n in H_U, there exists labeled ladder tree l_n^w of degree n in $H_{CK}(A)$ such that $\pi(l_n^w) = w$. Define a new map $Z_u : H_U \longrightarrow QSYM$ such that it maps each element $w \in H_U$ to

$$Z_u(w) := Z^\star(l_n). \tag{4.17}$$

It is observed that Z_u is a unique homomorphism of Hopf algebras. With the help of 3.1 and 4.7, one can define homomorphisms $\theta_1 := \beta_4 \circ \beta_3 = \beta_2 \circ \beta_1$ from $NSYM$ to H_U and $\theta_2 := \beta_4 \circ \alpha_4^\star = \beta_2 \circ \alpha_2^\star$ from H_{GL} to H_U. On the other hand, the surjective morphism π determines a new homomorphism ρ from H_{CK} to H_U such that for each unlabeled forest u in H_{CK}, it is defined by

$$\rho(u) := \sum_{v \in [u]} \pi(v). \tag{4.18}$$

With notice to the homomorphisms (3.4) and (4.17), the following commutative diagram can be obtained.

$$\begin{tikzcd}
 & SYM \arrow[dl, "\alpha_4"'] \arrow[dd, "\beta_4"] & \\
H_{CK} \arrow[dr, "\rho"] & & H_{GL} \arrow[ul, "\alpha_4^\star"'] \arrow[dl, "\theta_2"] \\
 & H_U \arrow[dl, "Z_u"'] & \\
QSYM & & NSYM \arrow[ul, "\theta_1"'] \arrow[dl, "\alpha_3"] \\
 & SYM \arrow[ul, "Z_u \circ \beta_4"] \arrow[uu, "\beta_4"] &
\end{tikzcd} \tag{4.19}$$

The dual of the maps Z_u and ρ are given by

$$Z_u^\star : NSYM \longrightarrow U(L_{\mathbb{U}})$$

$$Z_u^\star(z_n) := e_{-n}, \tag{4.20}$$

$$\rho^\star : U(L_{\mathbb{U}}) \longrightarrow H_{GL}$$

$$\rho^\star(e_{-n}) := l_n. \tag{4.21}$$

With the help of these maps and with notice to 4.7 and the given diagrams in 3.1, one can obtain a dual version of the diagram (4.19) given by

$$\begin{tikzcd}
 & SYM \arrow[dr, "\alpha_4"] & \\
H_{GL} \arrow[ur, "\alpha_4^\star"] & & H_{CK} \\
 & U(L_{\mathbb{U}}) \arrow[ul, "\rho^\star"'] \arrow[uu, "\beta_4^\star"'] \arrow[ur, "\theta_2^\star"'] \arrow[dr, "\theta_1^\star"] \arrow[dd, "\beta_4^\star"] & \\
NSYM \arrow[ur, "Z_u^\star"'] \arrow[dr, "(Z_u\circ\beta_4)^\star"'] & & QSYM \\
 & SYM \arrow[ur, "\alpha_3^\star"'] &
\end{tikzcd} \tag{4.22}$$

Graded dual relation for the pair $(H_{CK}(A), H_{GL}(A))$ and also the pair $(H_U, U(L_{\mathbb{U}}))$ provide another homomorphisms from H_U to $H_{GL}(A)$ and also from $H_{CK}(A)$ to $U(L_{\mathbb{U}})$.

Lemma 4.8. *Let H_1 and H_2 be graded connected locally finite Hopf algebras which admit inner products $(.,.)_1$ and $(.,.)_2$, respectively. If they are dual to each other, then there is a linear map $\lambda : H_1 \longrightarrow H_2$ such that*

- *λ preserves degree,*
- *For each $h_1, h_2 \in H_1 : (h_1, h_2)_1 = (\lambda(h_1), \lambda(h_2))_2$,*
- *For each $h_1, h_2, h_3 \in H_1$:*

$$(h_1 h_2, h_3)_1 = (\lambda(h_1) \otimes \lambda(h_2), \Delta_2(\lambda(h_3)))_2,$$

$$(h_1 \otimes h_2, \Delta_1(h_3))_1 = (\lambda(h_1)\lambda(h_2), \lambda(h_3))_2.$$

This linear map determines an isomorphism $\tau : H_2 \longrightarrow H_1^\star$ such that for each $h_1 \in H_1$ and $h_2 \in H_2$, it is defined by

$$< \tau(h_2), h_1 >:= (h_2, \lambda(h_1))_2.$$

[17]

Since $H_{CK}(A)$ and $H_{GL}(A)$ are graded dual to each other, therefore by 4.8 one can find a linear map $\lambda : H_{CK}(A) \longrightarrow H_{GL}(A)$ with the mentioned properties such that it defines an isomorphism τ_1 from $H_{GL}(A)$ to $H_{CK}(A)^\star$ given by

$$< \tau_1(t), s >:= (t, \lambda(s)) \tag{4.23}$$

where for rooted trees t_1, t_2, if $t_1 = t_2$ then $(t_1, t_2) = |sym(t_1)|$ and otherwise it will be 0. For each word w with length n in H_U, there is a labeled ladder tree l_n^w in $H_{CK}(A)$ such that $\pi(l_n^w) = w$. Now one can define a new homomorphism F given by

$$F : H_U \longrightarrow H_{GL}(A)$$

$$F(w) := \tau_1^{-1}((l_n^w)^\star). \tag{4.24}$$

Since H_U and $U(L_{\mathbb{U}})$ are graded dual to each other, therefore with the help of 2.1 and 4.8 one can obtain a linear map $\theta : H_U \longrightarrow U(L_{\mathbb{U}})$ with the mentioned

properties and after that an isomorphism τ_2 from $U(L_{\mathbb{U}})$ to $H_U^\star$ will be determined. For each element $x \in U(L_{\mathbb{U}})$ and word $w \in H_U$, it is given by

$$< \tau_2(x), w >:= (x, \theta(w)) \tag{4.25}$$

such that $(.,.)$ is the natural pairing on $U(L_{\mathbb{U}})$. For a labeled forest u, $\pi(u)$ is an element in H_U. By the natural pairing (given in 2.1), the dual of F is identified by

$$F^\star : H_{CK}(A) \longrightarrow U(L_{\mathbb{U}})$$

$$F^\star(u) := \tau_2^{-1}((\pi(u))^\star). \tag{4.26}$$

There is an interesting strategy to find a rooted tree representation from the universal affine group scheme $\mathbb{U}$ by the theory of construction of a group from an operad. Let P be an augmented set operad and $\mathbb{K}A_P = \bigoplus_n \mathbb{K}(P_n)_{\mathbb{S}_n}$ be the direct sum of its related coinvariant spaces with the completion $\widehat{\mathbb{K}A_P} = \prod_n \mathbb{K}(P_n)_{\mathbb{S}_n}$. There is an associative monoid structure on $\widehat{\mathbb{K}A_P}$. Let $\mathcal{G}_P$ be the set of all elements of $\widehat{\mathbb{K}A_P}$ whose first component is the unit $\mathbf{1}$. It is a subgroup of the set of invertible elements. In a more general setting, there is a functor from the category of augmented operads to the category of groups. For the operad P, we have a commutative Hopf algebra structure on $\mathbb{K}[\mathcal{G}_P]$ given by the set of coinvariants of the operad. It is a free commutative algebra of functions on $\mathcal{G}_P$ generated by the set $(g_\alpha)_{\alpha \in A_P}$. Each $f \in \mathcal{G}_P$ can be represented by a formal sum $f = \sum_{\alpha \in A_P} g_\alpha(f)\alpha$ such that $g_{\mathbf{1}} = 1$. There is a surjective morphism

$$\eta : g_\alpha \longmapsto \frac{F_{[\alpha]}}{|Aut(\alpha)|} \tag{4.27}$$

from the Hopf algebra $\mathbb{K}[\mathcal{G}_P]$ to the incidence Hopf algebra H_P such that it shows that at the level of groups, the affine group scheme $G_P(\mathbb{K})$ of H_P is a subgroup of the group $\mathcal{G}_P$. [4, 33]

Corollary 4.9. *(i) There is a surjective morphism from the Hopf algebra $\mathbb{K}[\mathcal{G}_{NAP}]$ to the Connes-Kreimer Hopf algebra H_{CK} of rooted trees.*

(ii) The affine group scheme $G_{CK}(\mathbb{K})$ of H_{CK} is a subgroup of $\mathcal{G}_{NAP}$.

Proof. By 3.3, (4.27), $\rho \circ \eta : \mathbb{K}[\mathcal{G}_{NAP}] \longrightarrow H_{CK}$ is a surjective morphism of Hopf algebras and therefore in terms of groups we will get the second claim. □

For the operad NAP, $\mathcal{G}_{NAP}$ is a group of formal power series indexed by the set of unlabeled rooted trees and there is also an explicit picture from the elements of $G_{NAP}(\mathbb{K})$ such that one can lift it to the universal Hopf algebra of renormalization.

Corollary 4.10. *(i) Universal affine group scheme $\mathbb{U}(\mathbb{C})$ is a subgroup of $\mathcal{G}_{NAP}$ and therefore each of its elements will be representable by a formal power series indexed with Hall rooted trees.*

(ii) An element $F = \sum_t g_t(F)t$ of $\mathcal{G}_{NAP}$ is in $\mathbb{U}(\mathbb{C}) \Longleftrightarrow$

- t is a Hall tree in $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$, (It does not belong to the Hopf ideal $I = ker\pi$.)

- And if $t = B^+_{f_n}(u)$ for some $u = t_1...t_k$ such that $t_1, ..., t_k \in \mathbf{H}(\mathbf{T}(\mathbf{A}))_U$ and $f_n \in A$, one has

$$g_t(F) = \prod_{i=1}^{k} g_{B^+_{f_n}(t_i)}(F).$$

Proof. One can extend the morphism (4.27) to the level of the decorated Hopf algebras and therefore by 4.6 and 4.9, there is a surjective map from $\mathbb{C}[\mathcal{G}_{NAP}](A)$ to H_U. It shows that in terms of groups, $\mathbb{U}(\mathbb{C})$ is a subgroup of $\mathcal{G}_{NAP}$. For the second case, according to the lemma 6.12 in [4], each element of $\mathcal{G}_{NAP}$ is in the subgroup G_{NAP} if and only if for each tree $t = B^+(t_1...t_k)$, we have the following condition

$$|sym(t)|g_t(F) = \prod_{i=1}^{k} |sym(B^+(t_i))|g_{B^+(t_i)}(F).$$

By 3.3, 4.1, 4.6, (4.27), 4.9 and since Hall trees have no symmetry, the proof is complete. □

Let $H(A)$ be a Hall set and $A^\star := \{a^\star : a \in A\}$. For each Hall word w, its associated *Hall polynomial* p_w in the free Lie algebra $\mathcal{L}(A^\star)$ is given as follows:

- If $a \in A$, then $p_a = a^\star$,
- If w is a Hall word of length ≥ 2 such that its corresponding Hall tree t_w has the standard decomposition (t_{w_1}, t_{w_2}), then $p_w = [p_{w_1}, p_{w_2}]$.

By induction, we can show that each p_w is an homogeneous Lie polynomial of degree equal to the length of w and also it has the same partial degree with respect to each letter as w.

In general, for each Hall set H, Hall polynomials form a basis for the free Lie algebra (viewed as a vector space) and their decreasing products $p_{h_1}...p_{h_n}$ such that $h_i \in H$, $h_1 > h_2 > ... > h_n$, form a basis for the free associative algebra (viewed as a vector space) [30]. About Hopf algebra H_U, we identified a Hall set $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$ such that $A = \{f_n\}_{n \in \mathbb{N}_{>0}}$ and for each f_n its associated Hall polynomial is given by $p_{f_n} = e_{-n}$.

Corollary 4.11. *(i) As a vector space, Hall polynomials associated to the Hall set* $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$ *form a basis for the Lie algebra* $L_{\mathbb{U}}$.

(ii) As a vector space, decreasing products of Hall polynomials with respect to the associated Hall words to the Hall set $\mathbf{H}(\mathbf{T}(\mathbf{A}))_U$ *form a basis for the free algebra* H_U.

In the next section, we focus on the essential importance of H_U in the mathematical reconstruction of perturbative renormalization and with the help of its new picture (given by Hall rooted trees), we are going to make a new representation from some important physical information such as counterterms.

5. Rooted tree representation of the universal singular frame

The Riemann-Hilbert correspondence consists of describing a certain category of equivalence classes of differential systems though a representation theoretic datum. For a given renormalizable QFT Φ with the related Hopf algebra H and affine group scheme G, one can identify a category of classes of flat equisingular $G-$connections. It is a neutral Tannakian category and therefore it should be equivalent with the category $\mathfrak{R}_{G^\star}$ (such that $G^\star := G \rtimes \mathbb{G}_m$) of finite dimensional linear representations of the affine group scheme of automorphisms of the fiber functor of the main category. It is reasonable to formulate the Riemann-Hilbert correspondence in a universal setting by constructing the universal category $\mathcal{E}$ of equivalence classes of all flat equisingular vector bundles. This category can cover the corresponding categories of all renormalizable theories and it means that when we are working on the

theory Φ, we should consider the subcategory $\mathcal{E}^{\Phi}$ of those flat equisingular vector bundles which are equivalent to the finite dimensional linear representations of $G^{\star}$. Since $\mathcal{E}$ is also a neutral Tannakian category, it would be equivalent to a category of finite linear representations of one special affine group scheme (related to the universal Hopf algebra of renormalization), namely universal affine group scheme $\mathbb{U}^{\star}$. In this case the fiber functor is given by $\varphi : \mathcal{E} \longrightarrow \mathcal{V}_{\mathbb{C}}$, $\Theta \longmapsto V$. [5, 6, 7]

From this correspondence, one specific element will be determined namely, the loop *universal singular frame* $\gamma_{\mathbb{U}}$ with values in $\mathbb{U}$. Because of the equivalence relation between loops (with values in the affine group scheme G) and elements of the Lie algebra (corresponding to G) ([5, 7]), for the presentation of the universal singular frame, we should identify suitable element from the Lie algebra $L_{\mathbb{U}}$ such that it is $e = \sum_{n \geq 1} e_{-n}$ (i.e. the sum of the generators of the Lie algebra). Since the universal Hopf algebra of renormalization is finite type, whenever we pair e with an element of the Hopf algebra, it would be only a finite sum and hence e does make sense. With help of the given rooted tree version of H_U in previous part, we are going to make a new representation from the universal singular frame based on rooted trees.

Theorem 5.1. *(i) e is an element of $L_{\mathbb{U}}$.*

(ii) $e : H_U \longrightarrow \mathbb{K}[t]$ is a linear map. Its affine group scheme level namely, $\mathbf{rg}$ *: $\mathbb{G}_a(\mathbb{C}) \longrightarrow \mathbb{U}(\mathbb{C})$ is a morphism that plays an essential role to obtain the renormalization group.*

(iii) The universal singular frame is given by $\gamma_{\mathbb{U}}(z, v) = Te^{-\frac{1}{z}\int_0^v u^Y(e)\frac{du}{u}}$.

(iv) For each loop $\gamma(z)$ in $Loop(G(\mathbb{C}), \mu)$, with help of the associated representation $\rho : \mathbb{U} \longrightarrow G$, the universal singular frame $\gamma_{\mathbb{U}}$ maps to the negative part $\gamma_-(z)$ of the Birkhoff decomposition of $\gamma(z)$ and also the renormalization group F_t in $G(\mathbb{C})$ is obtained by $\rho \circ$ $\mathbf{rg}$. [5, 6]

For a given smooth manifold M, let $C(\mathbb{R}^+)$ be the ring of piecewise real valued continuous functions on $\mathbb{R}^+$ and $\{X_a\}_{a \in A}$ be a family of smooth vector fields on M. Suppose $\mathcal{A}$ be the algebra over $C(\mathbb{R}^+)$ of linear operators on $C^{\infty}(M)$ generated by the vector fields X_a $(a \in A)$. For a family $\{g_a\}_{a \in A}$ of elements in $C(\mathbb{R}^+)$, set

$$X(x) = \sum_{a \in A} g_a(x) X_a. \tag{5.1}$$

It can be expanded as a series of linear operators in $\mathcal{A}$ of the form $\sum_w g_w X_w$ such that

- $w = a_1 ... a_m$ is a word on A,
- $X_1 = Id$ (identity operator), $X_w = X_{a_1} ... X_{a_m}$,
- $g_w = \int_{a_m} ... \int_{a_1} 1_{C(\mathbb{R}^+)}$ where each $\int_{a_i} : C(\mathbb{R}^+) \longrightarrow C(\mathbb{R}^+)$, $(1 \leq i \leq m)$ is a linear endomorphism defined by

$$\{\int_{a_i} g\}(x) := \int_0^x g(s) g_{a_i}(s) ds. \tag{5.2}$$

Generally, for the given associative algebra $\mathcal{A}$ over the commutative ring $\mathbb{K}$ generated by the elements $\{E_a : a \in A\}$, all elements in $\mathcal{A}$ are identified by formal series $\sum_w \mu_w E_w$ such that $\mu_w \in \mathbb{K}$. If $\mathcal{A}$ is a free algebra, then this representation will be unique. For the Hall set $\mathbf{H}(\mathbf{T}(\mathbf{A}))$ of labeled rooted trees with the corresponding Hall forest $\mathbf{H}(\mathbf{F}(\mathbf{A}))$, one can assign elements $E(u)$ given by

- $E(\mathbb{I}) = \mathfrak{e}$ (the unit of the Lie algebra $\mathfrak{a}$ of $\mathcal{A}$),
- For each Hall tree t with the standard decomposition $(t^1, t^2) \in \mathbf{H}(\mathbf{T}(\mathbf{A})) \times \mathbf{H}(\mathbf{T}(\mathbf{A}))$,

$$E(t) = [E(t^2), E(t^1)] = E(t^2)E(t^1) - E(t^1)E(t^2), \tag{5.3}$$

- For each $u \in \mathbf{H}(\mathbf{F}(\mathbf{A})) - \mathbf{H}(\mathbf{T}(\mathbf{A}))$ with the standard decomposition $(u^1, u^2) \in \mathbf{H}(\mathbf{F}(\mathbf{A})) \times \mathbf{H}(\mathbf{T}(\mathbf{A}))$,

$$E(u) = E(u^2)E(u^1). \tag{5.4}$$

Lemma 5.2. *(i) The Lie algebra $\mathfrak{a}$ is spanned by $\{E(t) : t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))\}$. It is called Hall basis.*

(ii) $\mathcal{A}$ is spanned by $\{E(u) : u \in \mathbf{H}(\mathbf{F}(\mathbf{A}))\}$. It is called PBW basis. [30]

Proposition 5.3. *For the locally finite total order set $\{f_n\}_{n\in\mathbb{N}}$, the universal singular frame is represented by*

$$\gamma_U(-z, v) = \sum_{n\geq 0, k_j>0} \alpha^U_{f_{k_1} f_{k_2} ... f_{k_n}} p_{f_{k_1}} ... p_{f_{k_n}} v^{\sum k_j} z^{-n}$$

such that $p_{f_{k_j}}$s are Hall polynomials.

Proof. By 5.1, we know that

$$\gamma_{\mathbb{U}}(z, v) = Te^{-\frac{1}{z}\int_0^v u^Y(e)\frac{du}{u}}.$$

After the application of the time ordered exponential, we have

$$\gamma_{\mathbb{U}}(-z, v) = \sum_{n\geq 0, k_j>0} \frac{e_{-k_1} ... e_{-k_n}}{k_1(k_1 + k_2)...(k_1 + ... + k_n)} v^{\sum k_j} z^{-n}. \tag{5.5}$$

In this expansion, the coefficient of the term $e_{-k_1}...e_{-k_n}$ is given by the iterated integral

$$\int_{0\leq s_1 \leq ... \leq s_n \leq 1} s_1^{k_1-1} ... s_n^{k_n-1} ds_1 ... ds_n. \tag{5.6}$$

By 4.3, one can see H_U as a free polynomial algebra on the Lyndon words on the set $\{f_n\}_{n\in\mathbb{N}_{>0}}$. Consider formal series

$$E := f_{k_1} + x f_{k_2} + x^2 f_{k_3} + ... \tag{5.7}$$

where

$$\mu_{k_j}(x) = x^{k_j - 1}. \tag{5.8}$$

By (5.2), for the variables $0 \leq s_1 \leq ... \leq s_n \leq 1$, we have

$$\{\int_{k_j} 1\}(s_j) = \int_0^{s_j} x^{k_j-1} dx. \tag{5.9}$$

For each word $f_{k_1} f_{k_2} ... f_{k_n}$, we can define the following well-defined iterated integral

$$\alpha^U_{f_{k_1} f_{k_2} ... f_{k_n}} := \int_{k_n} ... \int_{k_1} 1. \tag{5.10}$$

It is easy to see that the above integral is agree with the iterated integral associated to the coefficient of the term $e_{-k_1}...e_{-k_n}$. From the equations (2.15) and (5.5), the proof is complete. □

Proposition 5.3 determines uniquely a real valued map on the set $\mathbf{F}(\mathbf{A})$.

Definition 5.4. Let $A=\{f_n\}_{n\in\mathbb{N}}$ be the locally finite total order set corresponding to the universal Hopf algebra of renormalization. For the given map in 5.3 that associates to each word $w=f_{k_1}f_{k_2}...f_{k_n}$ a real value α^U_w and with notice to (4.6), define a new map α^U on $\mathbf{F}(\mathbf{A})$ such that

- $\mathbb{I}\longmapsto \alpha^U(\mathbb{I})=1$,
- For each non-empty labeled forest u in $\mathbf{F}(\mathbf{A})$,

$$\alpha^U(u)=\sum_{>_{\mathrm{u}(A)}}\alpha^U_{w(>_{\mathrm{u}(A)})}.$$

It is observed that for labeled rooted trees $t_1,...,t_m\in\mathbf{T}(\mathbf{A})$ and $f_{k_j}\in A$,

$$\alpha^U(t_1...t_m)=\alpha^U(t_1)...\alpha^U(t_m),\quad \alpha^U(B^+_{f_{k_j}}(t_1...t_m))=\int_{k_j}\alpha^U(t_1)...\alpha^U(t_m), \tag{5.11}$$

The map α^U (determined by the universal singular frame), with the above properties, is uniquely characterized.

With notice to (4.6), for a given map $\alpha:\mathbf{F}(\mathbf{A})\longrightarrow\mathbb{K}$, if $\alpha(\mathbb{I})=0$, then the *exponential map* is defined by

- *exp* $\alpha(\mathbb{I})=1$,
- For each $u\in\mathbf{F}(\mathbf{A})-\{\mathbb{I}\}$,

$$exp\ \alpha(u)=\sum_{k=1}^{|u|}\frac{1}{k!}\alpha^k(u), \tag{5.12}$$

and if $\alpha(\mathbb{I})=1$, then the *logarithm map* is defined by

- *log* $\alpha(\mathbb{I})=0$,
- For each $u\in\mathbf{F}(\mathbf{A})-\{\mathbb{I}\}$,

$$log\ \alpha(u)=\sum_{k=1}^{|u|}\frac{(-1)^{k+1}}{k}(\alpha-\epsilon)^k(u) \tag{5.13}$$

such that $\epsilon(\mathbb{I})=1$ and for $u\in\mathbf{F}(\mathbf{A})-\{\mathbb{I}\}$, $\epsilon(u)=0$. [27]

With help of the proposition 7 in [27], for the defined map 5.4 with the properties (5.11), there exists a real valued map β^U on $\mathbf{F}(\mathbf{A})$ given by

$$\alpha^U=exp\ \beta^U \tag{5.14}$$

such that for each $u\in\mathbf{F}(\mathbf{A})-\mathbf{T}(\mathbf{A})$,

$$\beta^U(u)=0. \tag{5.15}$$

Proposition 5.5.

$$\sum_{k_j>0,n\geq 0}\alpha^U_{f_{k_1}f_{k_2}...f_{k_n}}f_{k_1}f_{k_2}...f_{k_n}=exp\ (\sum_{t\in\mathbf{H}(\mathbf{T}(\mathbf{A}))_U}\beta^U(t)E(t))$$

such that

- $\{E(t):t\in\mathbf{H}(\mathbf{T}(\mathbf{A}))_U\}$ *(the set of all Hall polynomials) is the Hall basis for* $L_{\mathbb{U}}$*,*
- $\{E(u):u\in\mathbf{H}(\mathbf{F}(\mathbf{A}))_U\}$ *(the set of decreasing products of Hall polynomials) is the PBW basis for* H_U*.*

Proof. It is observed that for a given map $\alpha:\mathbf{F}(\mathbf{A})\longrightarrow\mathbb{K}$, if $\alpha(\mathbb{I})=0$, then

$$exp\ (\sum_w\alpha_wE_w)=\sum_{u\in\mathbf{H}(\mathbf{F}(\mathbf{A}))}exp\ \alpha(u)E(u), \tag{5.16}$$

and if $\alpha(\mathbb{I}) = 1$, then

$$log\ (\sum_{w} \alpha_w E_w) = \sum_{u \in \mathbf{H}(\mathbf{F}(\mathbf{A}))} log\ \alpha(u)E(u). \tag{5.17}$$

Now by (5.11) and with help of the continuous BCH formula (44) in [27], for each word $w = f_{k_1} f_{k_2} ... f_{k_n}$ on the set A, one can have

$$exp\ (\sum_{t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))_U} \beta^U(t)E(t)) = \sum_{w} \alpha_w^U w. \tag{5.18}$$

Hall basis and PBW basis related to H_U are determined with notice to 4.11 and 5.2. □

With notice to 5.3, 5.4 and 5.5, a Hall rooted tree representation from $\gamma_{\mathbb{U}}$ is obtained.

Definition 5.6. The formal series $\sum_{t \in \mathbf{H}(\mathbf{T}(\mathbf{A}))_U} \beta^U(t)E(t)$ (given by 5.5) is called *Hall polynomial representation* of the universal singular frame.

There is also another way to show that the universal singular frame has a rooted tree representation. 4.6 and (4.27) provide a surjective morphism from $\mathbb{C}[\mathcal{G}_{NAP}](A)$ to the Hopf algebra H_U and also corollary 4.10 shows that the universal affine group scheme $\mathbb{U}(\mathbb{C})$ is a subgroup of $\mathcal{G}_{NAP}$. Since $\gamma_{\mathbb{U}}$ is a loop with values in $\mathbb{U}(\mathbb{C})$, therefore for each fixed values z, v, $\gamma_{\mathbb{U}}(z, v)$ should be a formal power series of Hall rooted trees with the given conditions in 4.10.

For each loop in $Loop(G(\mathbb{C}, \mu))$, the universal singular frame maps to negative part of the Birkhoff decomposition of the given loop such that it determines counterterms, renormalization group and $\beta-$function of the theory. With notice to this fact and with help of the introduced representation by 5.1 (related to the given loop), one can map the Hall polynomial representation of the universal singular frame to these physical information and it means that with the help of formal sums of Hall trees (Hall forests) and Hall polynomials associated to the universal Hopf algebra of renormalization, one can obtain new representations from these physical information of the theory. This possibility is another reason for the universality of H_U.

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